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Comments on the Martingale Convergence Theorem.*)

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Let $(\Omega, \mathcal{B}, P)$ be a probability space and let X be a Banach space. A sequence of X-valued Bochner-integrable random variables f_n on Ω will be said to form a martingale with respect to the sub algebras $\mathfrak{A}_n$ $n=1,2,\dots$, $\mathfrak{A}_n \subset \mathfrak{A}_{n+1}$ (in short f_n, $\mathfrak{A}_n$ is a martingale) if

$$E^{\mathfrak{A}_n} f_{n+1} = f_n \qquad n \geq 1,$$

where $E^{\mathfrak{A}_n}$ is the conditional expectation operator with respect to the σ-algebra $\mathfrak{A}_n$. It is known that these operators are well-defined for arbitrary Banach-space-valued integrables functions. In the following it will be assumed that the algebra $\mathfrak{A} = \bigcup_{n=1}^{\infty} \mathfrak{A}_n$ generates the σ-algebra $\mathcal{B}$. The general case can be handled via standard reduction to this case.

My main concern will be proving almost everywhere (a.e.) convergence theorems for martingales. For the sake of brevity, I shall limit myself in this talk to considering only the following statements:

(S_1): If $f_n = E^{\mathfrak{A}_n} f$ then $\lim_{n\to\infty} f_n = f$ a.e. (strong limit in X)

(S_2): If $\{f_n, \mathfrak{A}_n\}$ is a martingale and the f_n's are uniformely integrable (i.e. $\lim_{N\to\infty} \int \|f_n\| \cdot I\{\|f_n\| > N\}\, dP = 0$ uniformely in $n \geq 1$) then $\exists f_\infty$ such that $\lim_{n\to\infty} f_n = f_\infty$ a.e.

(S_3): If $\{f_n, \mathfrak{A}_n\}$ is a martingale with $\sup_{n\geq 1} E(\|f_n\|) < \infty$ then $\exists f_\infty$ such that $\lim_{n\to\infty} f_n = f_\infty$ a.e.

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In § 2 I shall prove that (S_1) is always true. In this generality, the result is proved by other methods in Chatterji (2b) and also in A.I. and C.I. Tulcea (6). The present prove, paralleling the proof in the scalar-valued case as in Billingsley (1), is as simple (possibly, some would wish to say trivial) as one could wish for.

In § 3, I shall prove the main theorem of this paper viz, that if X satisfies the following (RN) condition (RN for Radon-Nikodym) then (S_3) (and hence trivially (S_2)) is valid for all martingales. The condition referred to is:

(RN): Every σ-additive X-valued set function μ on $\mathcal{B}$ of bounded variation with the property that V_μ , the variation of μ , is absolutely continuous with respect to P$(V_\mu \ll P)$ can be represented as the indefinite integral of a X-valued Bochner-integrable function. The non-negative measure V_μ is defined as follows:

$$V_\mu(A) = \sup\left\{\sum_{i=1}^{n} \|\mu(A_i)\| \ : A_i A_j = \phi \ , A_i \in \mathcal{B} \ , \bigcup_{i=1}^{n} A_i = A, \ n \geq 1\right\}$$

The implication (RN) $\Rightarrow$ (S_3) is more general than the statements obtainable from (6). It also follows that (S_3) is valid for reflexive X separable dual spaces X, statements explicitly made in (6). For reflexive X, (S_2) (weaker than (S_3)) was proved by different methods in (2a,b) and by Scalora (5). That some condition on X is necessary for the validity of (S_2) or (S_3) ist demonstrated by the counterexample in (2a). Here a martingale f_n is constructed which takes values in $L^1(0,1)$ and which does not converge in any sense, weak or strong, anywhere, althoug amongst other things, $\|f_n\| \equiv 1$ for all $n \geq 1$.

In § 4, it is shown that the (RN) condition is also necessary if $\mathcal{B}$ is seperable (generared by a denumberable class of subsets). More precisely, in this case (S_2), (S_3) and (RN) are equivalent conditions.

<u>§ 2:</u> The main probabilistic tool is the following lemma:

<u>Lemma 1:</u>

For any martingale $\{f_n, \mathcal{O}_n\}$, if $A \in \mathcal{O}_{n_o}$ and $\varepsilon > 0$ then

$$P\left\{A; \ \sup_{k \geq n_o} \| f_k \| > \varepsilon\right\} \leq \frac{1}{\varepsilon} \sup_{k \geq n_o} \int_A \| f_k \| \, dP$$

The lemma is known and an easy consequence of the fact that $\| f_n \|$ is a submartingale.

Theorem 1.

(S_1) : For any space X, $\lim_{n\to\infty} E^{\mathcal{A}_n} f = f$ a.e. (P).

Sketch of proof: If f is measurable $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ then (S_1) is trivial since $E^{\mathcal{A}_n} f=f$ for sufficiently large n. for a general f, $\exists$ f_ε measurable $\mathcal{A}$ such that

$$E (\| f - f \|) < \varepsilon$$

The following obvious inequality

$$\| E^{\mathcal{A}_n} f - E^{\mathcal{A}_m} f \| \leqslant \| E^{\mathcal{A}_n} f_\varepsilon - E^{\mathcal{A}_m} f \| + 2 \sup_{k\geqslant 1} E^{\mathcal{A}_k} \| f - f \|$$

coupled with lemma 1 leads us to (S_1) quite smoothly.

§ 3: Given a martingale $\{ f_n, \mathcal{A}_n \}$, define the set-functions μ_n on $\mathcal{A}_n$ as follows:

$$\mu_n (A) = \int_A f_n \, dP.$$

The martingale property is equivalent to the property that μ_{n+1} is an extension of μ_n to $\mathcal{A}_{n+1}$. Hence for any $A \in \mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$, $\lim_{n\to\infty} \mu(A) = \mu(A)$ exists. The set-function μ on $\mathcal{A}$ is an X-valued finitely additive set-function. The set-function is of bounded variation if f $\sup_{n\geqslant 1} \int \| f_n \| dP < \infty$. The main difficulty in proving martingale convergence theorems is that μ may not be σ -additive. The following lemma gives a way out.

Lemma 2:

Let P be a probability measure on the algebra $\mathcal{A}$ of subsets of a space Ω and μ a finitely additive X-valued set-function of bounded variation an $\mathcal{A}$. Then

$$\mu = \eta + \sigma$$

where η, σ are both of bounded variation and η is a finitely additive set-function such that V_η (the variation of η) is singular with respect to P (i.e. given ε, $\delta > 0$, $\exists A \in \mathcal{O}\!\iota$, $P(A) < \varepsilon$, $V_\eta(A') < \delta$) and σ is a σ-additive set-function such that V_σ is absolutely continuous with respect to P (i.e. given $\varepsilon > 0$, $\exists\ \delta > 0, P(A) < \delta \Rightarrow V_\sigma(A) < \varepsilon$)

The main idea behind the proof of the lemma will be sketched. One transfers P and μ to the space (S, Σ_1) where S is a totally disconnected compact Hausdorff space and Σ_1 is the algebra ofclopen sets in S, Σ_1 being isomorphic to $\mathcal{O}\!\iota$. It turns out that P and μ are σ-additive on Σ_1 and hence can be extended to the σ-algebra Σ_2 generated by Σ_1. (These are standard methods in this sort of work; see e.g. (3) pp.312-13). On these extended measures on Σ_2 apply the Lebesgue-decomposition theorem as proved by Rickart (4) and then retrace the way back through Σ_1 to $\mathcal{O}\!\iota$ to obtain the decomposition indicated in the lemma.

With the help of lemma 2, I shall now prove the main theorem of this talk:

Theorem 2:

If X satisfies the (RN) property with respect to P on $\mathcal{B}$ then any martingale $\{f_n, \mathcal{O}\!\iota_n\}$ with $\sup_{n \geq 1} \int \|f_n\| dP < \infty$ converges i.e. $\exists f_\infty$ such that

$$\lim_{n \to \infty} f_n = f_\infty \quad \text{a.s.} \quad (P)$$

Sketch of the proof: Let μ be as before and η, σ as in lemma 2. μ restricted to $\mathcal{O}\!\iota_n$ is an integral. σ, being absolutely continuous with resoect to P, is also an integral since X has the (RN) property.

Let $\sigma(A) = \int_A h \, dP \quad A \in \mathcal{O}\!\iota$ .And $\quad \sigma(A) = \sigma_n(A) = \int_A h_n \, dP \quad A \in \mathcal{O}\!\iota_n$

$$\text{Clearly} \quad h_n = E^{\mathcal{O}\!\iota_n} h \,.$$

Hence η restricted to $\mathcal{O}\!\iota_n$ is also an integral i.e.

$$\eta(A) = \eta_n(A) = \int_A g_n \, dP \qquad A \in \mathcal{O}\!\iota_n$$

In other words, $f_n = g_n + h_n$

where g_n, h_n are also martingales with respect to $\mathcal{O}\!\iota_n$.

Moreover h_n is $E^{\mathcal{O}_n} h$ and hence theorem 1 ensures the convergence of h_n to a limit.

I shall now show that $\lim g_n = 0$ a.s. (P).

Given $1 > \varepsilon$, $\delta > 0$, find $A \in \mathcal{O}$ (and hence $A \in \mathcal{O}_{n_o}$ for some n_o) such that
$P(A') + V_\eta(A) < \frac{\delta\varepsilon}{2}$.

Now

$$P\Big\{ \sup_{n \geq n_o} \| g_n \| > \varepsilon \Big\} = P\Big\{ A', \sup_{n \geq n_o} \| g_n \| > \varepsilon \Big\} + P\Big\{ A; \sup_{n \geq n_o} \| g_n \| > \varepsilon \Big\}$$

$$< \frac{\delta\varepsilon}{2} + \frac{1}{\varepsilon} \sup_{n \geq n_o} \int_A \| g_n \| \, dP \qquad \text{(by lemma 1)}$$

$$< \frac{\delta\varepsilon}{2} + \frac{1}{\varepsilon} V_\eta(A) < \frac{\delta\varepsilon}{2} + \frac{\delta}{2} < \delta .$$

This is clearly enough to show that $\lim g_n = 0$ a.s. (P).

§ 4. In this section, the main thing is the following lemma for real-valued submartingales:

Lemma 3:

If $\{ g_n, \mathcal{O}_n \}$ is a positive submartingale with $\sup_{n \geq 1} E(g_n) < \infty$ such that

$$\mu(A) = \lim_{n \to \infty} \int_A g_n \, dP, \quad A \in \mathcal{O} = \bigcup_{n=1}^{\infty} \mathcal{O}_n$$ is a σ - additive P-continuous set-function then g_n's are uniformely integrable.

Sketch of proof: If $g_n \geq 0$ is a martingale then it is easy. In general $\exists$ h_n a martingale so that $0 \leq g_n \leq h_n$ and such that $\{h_n\}$ induces the same μ. Hence the lemma.

Theorem 3:

If $\mathcal{B}$ is separable then $(S_2) \Rightarrow (RN)$. Hence in this case $(S_2) \Leftrightarrow (S_3) \Leftrightarrow (RN)$.

Sketch of proof: Let $\mathcal{B}$ be generated by $A_1, A_2, \ldots$ and $\mathcal{O}_n$ = the σ-algebra generated by $A_1 \ldots\ldots A_n$. Given a set-function μ on $\mathcal{B}$ satisfying the condition in (RN), the martingale $\{ f_n, \mathcal{O}_n \}$ induced by μ is such that $\| f_n \|$ satisfies the conditions of Lemma 3. Hence (S_2) implies the convergences of f_n to f_∞. From hereon it is trivial to show that μ is the indefinite integral of f_∞.

Note:

In the real-valued case the general martingale convergence theorem S_3 can be deduced rapidly from S_1 by the following sequence of arguments:

(I) $(S_1) \Rightarrow (S_2)$ because f_n uniformely integrable implies that $\exists\, h_k$ such that $f_{n_k} \longrightarrow f$ weakly in L^1 for some f.
Hence $E^{\mathfrak{A}_n} f_{n_k} \xrightarrow[k\to\infty]{} E^{\mathfrak{A}_n} f$ weakly. But $E^{\mathfrak{A}_n} f_{n_k} = f_n$ for large n_k. Hence $E^{\mathfrak{A}_n} f = f_n$ etc. Next

(II) every uniformely integrable submartingale converges: this follows from (i) via the Doob-decomposition for submartingales.

(III) Every positive martingale f_n converges since e^{-f_n} is a uniformely bounded semi-martingale.

(IV) An arbitrary martingale f_n with $\sup E\, f_n^+ < \infty$ converges because it is the difference of two positive martingales and (III).

From here the same theorem for submartingales can also be easily obtained.

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